The Moonshine Module for Conway's Group

John Duncan and Sander Mack-Crane*
Case Western Reserve University

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Moonshine

Moonshine is a series of connections

modular functions $\longleftrightarrow$ representation theory of finite groups

Moonshine has been discovered for the monster group $\mathbb{M}$, Conway's group Co_0, the Mathieu groups M_{24} and M_{12}, ...

We'll focus on moonshine for Conway's group.

Conway's Group

Conway's group Co_0 is the automorphism group of a 24-dimensional lattice known as the Leech lattice.

Co_0 has $8\,315\,553\,613\,086\,720\,000$ elements, and 167 irreducible representations of dimension

$$1,\ 24,\ 276,\ 299,\ 1771,\ 2024,\ 2576,\ 4576,\ 8855, \ldots.$$

Modular Functions

The upper half plane

$$\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$$

can realize a model of the hyperbolic plane, and the group of orientation-preserving isometries is $\mathrm{SL}_2 \mathbb{R}$ acting by linear fractional transformations.

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Given a discrete group $\Gamma < \mathrm{SL}_2\,\mathbb{R}$, we can form the orbit space $\Gamma \backslash \mathbb{H}$.

Then add finitely many points to obtain a compact surface $\Gamma \backslash \mathbb{H}^*$.

Modular Functions

For $\Gamma < \mathrm{SL}_2 \mathbb{R}$ a discrete subgroup, a *modular function* for Γ is a meromorphic function $\Gamma \backslash \mathbb{H}^* \to \mathbb{C}$.

The set of modular functions on Γ forms a field, and this field is generated by a single element exactly when the genus of $\Gamma \backslash \mathbb{H}^*$ is 0 (in this case the group Γ is said to have *genus* 0).

A generator is called a *principal modulus* (or Haputmodul) for Γ.

Modular Functions

Equivalently, a *modular function* for $\Gamma < \mathrm{SL}_2\,\mathbb{R}$ is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau) \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

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If $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, then $f(\tau + 1) = f(\tau)$ and

$$f(\tau) = \sum_{n \geq -N} a_n q^n \qquad\qquad (q = e^{2\pi i \tau}).$$

Principal moduli are not unique, but there is a unique *normalized* principal modulus for Γ with Fourier expansion $q^{-1} + 0 + O(q)$.

Modular Functions

Example: the group $\Gamma_0(2) < \mathrm{SL}_2\,\mathbb{R}$ consists of integer matrices of determinant 1 which are upper triangular mod 2.

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{Z}, ad - 2bc = 1 \right\}$$

$\Gamma_0(2)$ is a genus 0 group, and its normalized principal modulus is

$$f(\tau) = q^{-1} - 0 + 276q - 2048q^2 + 11202q^3 - \cdots$$

Moonshine

Representations of Co_0:

$$1,\ 24,\ 276,\ 299,\ 1771,\ 2024,\ 2576,\ 4576,\ 8855, \ldots$$

Normalized principal modulus for $\Gamma_0(2)$:

$$f(\tau) = q^{-1} - 0 + 276q - 2048q^2 + 11202q^3 - \cdots$$

Observation:

$$\begin{aligned} 1 &= 1 \\ 276 &= 276 \\ 2048 &= 2024 + 24 \\ 11202 &= 8855 + 2024 + 299 + 24 \\ &\vdots \end{aligned}$$

Moonshine

Conjecture

There is a graded representation

$$V = \bigoplus_{i \geq -1} V_i$$

of Co_0 such that

$$\dim V = \sum_{i \geq -1} \dim V_i \, q^i$$

is the normalized principal modulus of $\Gamma_0(2)$.

Moonshine

Conjecture

There is a graded representation

$$V = \bigoplus_{i \geq -1} V_i$$

of Co_0 such that

$$\mathrm{tr}_V\, g = \sum_{i \geq -1} \mathrm{tr}_{V_i}\, g\, q^i$$

is the normalized principal modulus of a genus 0 subgroup of $\mathrm{SL}_2\,\mathbb{R}$ for all $g \in Co_0$.

Construction

1. Let $\mathfrak{a} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ be a complex vector space enveloping the Leech lattice.

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3. In a similar way, construct the twisted vertex algebra module
$$A(\mathfrak{a})_{\mathrm{tw}} = A(\mathfrak{a})^0_{\mathrm{tw}} \oplus A(\mathfrak{a})^1_{\mathrm{tw}}.$$
4. Set
$$V^{s\natural} = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1_{\mathrm{tw}}.$$

This is a graded representation of Co_0.

Moonshine

Theorem (Duncan and M-C)

For all $g \in Co_0$,

$$\mathrm{tr}_{V^{s\natural}}\, g = \sum_{i \geq -1} \mathrm{tr}_{V_i^{s\natural}}\, g\, q^i$$

is the normalized principal modulus of a genus 0 subgroup of $\mathrm{SL}_2\,\mathbb{R}$.

Physics

The vertex algebra $V^{s\natural} = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1_{\mathrm{tw}}$ has a canonical vertex algebra module

$$V^{s\natural}_{\mathrm{tw}} = A(\mathfrak{a})^0_{\mathrm{tw}} \oplus A(\mathfrak{a})^1,$$

which is also a representation of Co_0.

We can introduce a bigrading

$$V^{s\natural}_{\mathrm{tw}} = \bigoplus_{i,j} V^{s\natural}_{\mathrm{tw},ij}$$

and the graded traces

$$\mathrm{tr}_{V^{s\natural}_{\mathrm{tw}}}\, g = \sum_{i,j} \mathrm{tr}_{V^{s\natural}_{\mathrm{tw},ij}}\, g\, q^i y^j$$

for $g \in Co_0$ (fixing a 4-dimensional sublattice in their action on the Leech lattice) are twined elliptic genera of non-linear sigma models on K3 surfaces.